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Arbitrage Equilibria in Large Games with Many Commodities*

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Abstract

Can identical goods sell at different prices in identical markets when people are perfectly mobile? We provide a formal account of strategic behaviour in large games with many commodities, and exhibit how it drives price dispersion at equilibrium. Interactions between agents are modelled using a Shapley–Shubik market game. We demonstrate the failure of the law of one price in this setup through a robust counterexample. The proposed model, and our findings, constitute an alternative and plausible explanation to some "anomalies" which routinely appear in a wide array of fields, ranging from banking, business economics, to international, and labour economics. *JEL Classification*: C72, D43, L1

Keywords: Oligopoly; Infinite-dimensional commodity space; Arbitrage; Strategic Behaviour

1 Introduction

The *Law of One Price* (LOP) is a fundamental concept underpinning almost all the subfields of economics and finance. It stipulates that there is a single price which clears all markets for a commodity at equilibrium. In a finite-commodity framework, Toraubally (2018) proved that even in frictionless markets featuring large numbers of agents—all of whom can arbitrage prices if they wish to—it is possible for the LOP to fail. In this paper, we go one (nontrivial) step further: we solve a longstanding open problem, and establish that the LOP may fail to obtain in large frictionless[1] markets with infinitely many commodities. Why this infinite-dimensional contribution is of paramount importance is best understood by considering the following points.

First, infinite-dimensional commodity models have become prominent in the social sciences because they capture natural aspects of the world that cannot be examined in their finite-dimensional counterparts. An obvious case in point is commodity differentiation. Taking online search markets and stock markets as the archetypal examples, there are inarguably only finitely many commodities which are actually traded by business executives, hedge-fund managers, investors, and other

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[1] "Frictionless" is to be understood in the sense that no good or money is lost or leaves the system.

customers. Yet, these same commodities are conceivably bought and sold in infinitely many variations. A theory which aims to explain such trades must no doubt employ infinite-dimensional analysis, where each variation can be regarded as a separate commodity. Second, models in fields such as business economics, and international trade typically feature large numbers of commodities and players. The failure of the LOP is well documented in these areas, and this paper aims to explain such price dispersions *solely* through strategic forces. Our motivation for pursuing this route is thus: in a paper which arguably encompasses both the above-mentioned categories, Haskel and Wolf (2001) find that for identical products—furniture, sold by IKEA—price deviations of 20%-50% exist. However, and crucially, this price disparity cannot be narrowed down to differences in distribution costs, local taxes, and even tariffs, which leads the authors to conclude there may be other, *strategic* influences at work.[2]

Consider, by way of additional motivation, the importance of using an infinite-dimensional framework in labour markets. Indubitably, the number of workers employed at any point in time is finite; nevertheless, there are likely infinitely many worker types. Here, the failure of the LOP also has important implications. Any entrepreneur, or executive of a state-owned organisation needs to know the ramifications of competing firms paying different, especially higher, wages to workers of the same skill type employed in identical jobs. As Ehrenberg and Smith (2012) argue, the basic labour market model built on the assumption of costless worker mobility between employers has an unequivocal prediction: similarly-skilled workers performing the same jobs in similar working conditions must receive the same wage. "If a firm currently paying the market wage were to ... pay even a penny less per hour, ... it would instantly lose all its workers to firms paying the going wage" (Ehrenberg and Smith, 2012, p. 129). However, as the authors remark, data from the US Bureau of Labor Statistics show that in 2009, registered nurses in Albany, Madison, and Sacramento—all medium-sized state capitals with very comparable costs of living—received mean hourly wages of \$28.87, \$33.79, and \$43.16, respectively. While the extant literature cannot explain this failure in the LOP without the introduction of economic frictions (see, e.g., Burdett and Mortensen, 1998, and Postel-Vinay and Robin, 2002, who analyse wage dispersion using labour-market models with search frictions), in this paper we provide an easily-adaptable theory to explain those wage differentials *uniquely* via a (frictionless) strategic-behavioural lens.

In this paper, we model strategic interactions between agents using a *strategic market game* (SMG)[3] with multiple markets per commodity. In an SMG, prices are determined by the buy-and-sell decisions of agents: increased buying and selling drive prices upward and downward, respectively. SMGs thus provide an elegant solution to an inescapable problem which executives and even pol-

[2] Elegant theories of equilibrium price dispersion have been proposed by, e.g., Burdett and Judd (1983), Salop and Stiglitz (1977), and Reinganum (1979), but these rely crucially on the existence of strictly positive search/information-gathering costs, as opposed to strategic considerations.

[3] SMGs à la Shapley–Shubik (1977) were proposed as a (noncooperative) general-equilibrium alternative to the Walrasian model of exchange, which relies on prices being given by Adam Smith's invisible hand. The multiple-market-per-commodity SMG was first analysed—albeit in an altogether different form to ours—in a seminal paper by Koutsougeras (2003), with only finitely many players. For an excellent overview of the wide-ranging applicability of SMGs, see Goenka (2003), Goenka et al. (1998), Xefteris and Ziros (2017), and the references therein.

icymakers face in their day-to-day jobs: modelling prices that depend in a reasonable way on individuals' trading decisions.[4] There are two types of agents, pure traders, and trading-post/market owners. The market owners, like the pure traders, buy and sell in the markets for each commodity. However, they also impose a proportional service charge on agents who trade on their platforms.[5] This charge is the *same* across *all* markets for the *same commodity*,[6] and is imposed only on the net trades of agents at a post.[7] It is taken to be exogenously given: assume that an outside agency/a government chooses and imposes this charge/tax before trading takes place. We show in this setup with infinitely many commodities that when agents are strategic in their decision-making process, the LOP can fail, even with many large and small agents.

It is perhaps helpful at this point to spell out some features of the game hereby proposed. In the multiple-market-per-commodity SMG that we consider, all traders are allowed to *simultaneously* trade in any number of commodities, and at however many trading posts for each commodity they would like to—i.e., given their financial constraints, agents have complete freedom with regard to which commodity(ies) to trade in, and which market(s) to place their orders at. Next, a well-known feature of Shapley–Shubik market games with numéraire is that if agents are *bindingly* financially-constrained at equilibrium, then a *true*[8] arbitrage opportunity could well and truly exist, but not be taken advantage of. In particular, agents would *want* to arbitrage prices, but would be *unable* to, simply because they lack cash. However, in this paper we present stronger results: agents face no binding constraints at equilibrium. They are entirely free, and also have the financial means to 'play' the prices (for any good) in an attempt to make a profit. Nonetheless, they interestingly still *choose* to stay put. Admittedly, at a glance, the notions of unequal market-clearing prices across identical platforms (with perfectly-mobile agents), and equilibrium seem incompatible. Yet, this unequal-price situation is tenable because any unilateral deviation from this "state of repose" leads agents to a worse payoff—the intricacies of this puzzle are dissected in Section 4.

It is not obvious that the violation of the LOP in large games with finitely many commodities applies to infinite-commodity games as well. This is because: (i) it is well known that in infinite-commodity economies, there exists a formidable amount of variation in agents' characteristics and strategy sets. If these characteristics are sufficiently dissimilar, then getting agents to achieve the required magnitude and direction of net trades for the LOP to fail can be truly very difficult; (ii) and as Kreps (1981), who considers a model with infinitely many commodities argues, it is both necessary

[4]This is especially relevant to the banking literature regarding issues associated with the "pricing" of loans: the so-called 'bad management' hypothesis—bank managers who do not practise adequate loan underwriting, monitoring, and control (see Izzeldin and Tsionas, 2018).

[5]In real-world markets, trading on privately-owned platforms is almost never free.

[6]This implies that markets for the same commodity are *identical* to traders. Yet, the LOP still fails. This means that once the service charge is accounted for, agents also pay different *effective* prices for the same commodity. Accordingly, the failure of the LOP as we hereby analyse is a much deeper and stronger concept than merely differing nominal prices.

[7]This is reminiscent of financial markets: brokerages charge a commission if an order is filled, but none if it is unfilled. We note that when a trader enters a market both as a buyer and seller, there are identical amounts of purchases and sales that cancel each other. Such "wash-sales" affect none of the market-clearing price and the resulting allocations, for all agents (see, e.g., Postlewaite and Schmeidler, 1978). It is only the "thickness" of that market which is affected. But then again, agents are free to choose which market(s) to trade at, such that being charged only on their net trades is natural.

[8]As opposed to the price inequalities in the present paper, which are but *illusions* of arbitrage opportunities.

and sufficient that there be no 'arbitrage opportunities' for an equilibrium to exist, where a bundle is 'priced by arbitrage' if it commands a *unique price* without creating an arbitrage opportunity; (iii) finally, generalising Toraubally's (2018) model to an infinite-commodity setup involves overcoming many mathematical difficulties which we next summarise. In Toraubally (2018), Banach spaces are used throughout. Hence, familiar concepts such as Bochner integrability and Fréchet differentiability can be routinely used. However, in the present study, we look at nonnormable locally convex topological vector spaces, giving rise to two issues. First, Bochner integrability can no longer be used. This makes proving the existence of measurable and (appropriately-) integrable strategy maps that take values into these spaces considerably more difficult. Second, Fréchet differentiability is no longer applicable, meaning more general concepts of differentiability have to be used, which in turn implies that extra care needs to be taken when composing functions.

2 The model

We consider a pure exchange economy with small agents, represented by an atomless continuum, and large agents, represented by atoms. We let the set of agents be denoted by $N = N_0 \cup A \cup C$, where $N_0 = (0,1]$, $A = \{2, ..., H\}$, and $C = \{H+g\}_{g=1}^{\infty}$. The space of agents is denoted by the measure space $(N, \mathcal{N}, \mu)$, where $\mathcal{N}$ is the collection of all μ-measurable sets of N, and μ is an extended real-valued, σ-additive measure defined on $\mathcal{N}$. Let $\mathcal{N}_{N_0}$, $\mathcal{N}_A$ and $\mathcal{N}_C$ denote the restricition of $\mathcal{N}$ to N_0, A, and C, respectively. We define μ to be the Lebesgue measure when restricted to $\mathcal{N}_{N_0}$, the counting measure when restricted to $\mathcal{N}_A$, and $\mu = \mu_C$ when restricted to $\mathcal{N}_C$, where μ_C is such that for each agent $(H+g) \in C, g \in \mathbb{N}$, $\mu_C(H+g) = \frac{1}{\pi^g}, \pi \in (1, \infty)$.

We denote the set of commodities *bought and sold* in this economy by $K = \{1, 2, 3, \ldots\}$. There is also a commodity, m, which in addition to yielding utility in consumption, acts as money. We define the consumption set by X, with a commodity bundle in X represented by $x = (x_k)_{k \in \{m\} \cup K}$.

There are two types of agents in this economy, pure traders, and post owners. Each pure trader $h \in N$ is characterised by a preference relation, which is representable by a utility function $u_h : X \to \mathbb{R}$, and an initial endowment of commodities $e(h)$. Each post owner $i \in N$ is characterised by a preference relation representable by a utility function $u_i : X \to \mathbb{R}$, and initial endowments of commodities $e(i)$, and trading posts $\Upsilon^i = \{\Upsilon^i_k\}_{k \in \mathbb{N}}$, where Υ^i_k denotes the post for k owned by i. W.l.o.g., we let post owners lie in A only, and assume that the capacity of each post is $\mu(N)$.[9]

Before trading starts, an outside agency allocates a service charge to post owners, which agents then take as given. This proportional service charge $c^k \in (0,1), k \in K$, is the same across all posts for a good k, but may differ across commodities—i.e., let $|P| < \infty$ denote the total number of (large) post owners, such that $c^{1,k} = \cdots = c^{|P|,k} = c^k$, and $c^{1,l} = \cdots = c^{|P|,l} = c^l$ for all $k, l \in K$, $k \neq l$, but c^k does not have to be equal to c^l.

Throughout this paper, we will employ the following assumptions:

[9]This way, post owners cannot prevent agents from trading on their platforms. Hence, all agents, large or small, are perfectly free to choose *whether*, and where to trade, and to arbitrage prices if they wish to.

Assumption (i). *The consumption set X is the positive cone of the space of all real-valued sequences, i.e., $X \coloneqq \{x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} : x_n \geq 0 \ \forall n \in \mathbb{N}\}$, where $\mathbb{R}^{\mathbb{N}}$ is endowed with the product topology τ_P.*

Assumption (ii). *For each $n \in N$, $e(n) = (\alpha_k)_{k \in \{m\} \cup K}$, where $0 < \alpha_k < \frac{\bar{\beta}}{\mu(N)}$ for every $k \in \{m\} \cup K$, and $0 < \bar{\beta} < \infty$.*

Assumption (iii). *Utility functions for all $n \in N$ are continuous in the product topology on X, strictly concave, continuously Gâteaux-differentiable in admissible directions, differentiably strictly monotone, and satisfy $\lim_{x_k \to 0} \frac{\partial u}{\partial x_k} = \infty, k \in \{m\} \cup K$.*

2.1 The strategic market game

Trade in the economy is organised through a system of trading posts (markets), at which agents offer commodities for sale, and place bids for purchases of commodities. Bids, b, for commodities $k \in K = \{1, 2, \ldots\}$, are placed in terms of commodity m, while sales, q, are made in terms of commodities $1, 2, \ldots$. Define $Y = (\mathbb{R}^{\mathbb{N}}, \tau_P) \times (\mathbb{R}^{\mathbb{N}}, \tau_P)$—i.e., each factor space $\mathbb{R}^{\mathbb{N}}$ is endowed with the product topology, and Y is supplied with the $\tau_P \times \tau_P$ product topology. Consider the subset $\mathbb{R}^{\mathbb{N}}_+ \times \mathbb{R}^{\mathbb{N}}_+$ of Y, and denote by Θ the space $(\mathbb{R}^{\mathbb{N}}_+, \tau_S) \times (\mathbb{R}^{\mathbb{N}}_+, \tau_S)$, where $\tau_S \coloneqq \{\mathbb{R}^{\mathbb{N}}_+ \cap \mathcal{O} : \mathcal{O} \in \tau_P\}$ and Θ is given the $\tau_S \times \tau_S$ product topology. Thus, agents' strategy sets can be described by a measurable correspondence $S : N \rightrightarrows 2^{\Theta}$, such that

$$S(n) = \Big\{(b(n), q(n)) \in \Theta : \sum_{k=1}^{\infty} \sum_{s=1}^{|P|} b_k^s\,(n) + \Lambda(n) \leq e_m\,(n)\,; \sum_{s=1}^{|P|} q_k^s\,(n) \leq e_k\,(n)\,, k \in K\Big\},$$

where $\varphi_k^s(n)$, $\varphi = b, q$, denotes the strategies of agent $n \in N$ at the post owned by $s \in A$ for commodity k, and $\Lambda(n)$ is the total premium payable by n (how $\Lambda(n)$ is calculated is found below).

A strategy profile consists of a pair of measurable mappings $b : N \to (\mathbb{R}^{\mathbb{N}}_+, \tau_S)$ and $q : N \to (\mathbb{R}^{\mathbb{N}}_+, \tau_S)$ such that $(b(n), q(n)) \in S(n)$ $a.e$ in N, i.e., a strategy profile is a measurable choice from the graph of the correspondence S, $Gr(S)$. We next derive the following intermediate results, which are themselves of independent interest:

PROPOSITION 1. The correspondence S has measurable graph.

PROPOSITION 2. The measurable mappings b and q exist.

PROPOSITION 3. The maps b and q are Gelfand–Pettis-integrable, and also weakly measurable.

In view of Propositions 1, 2 and 3, for a given strategy profile $(b, q) \in Gr(S)$, we may then define $B_k^s = \int_N b_k^s(n) d\mu < \infty$, and $Q_k^s = \int_N q_k^s(n) d\mu < \infty$. Transactions at each post clear through the price $p_k^s = B_k^s / Q_k^s$. For $k \in K$, we let $z_k^s(n) = b_k^s(n)/p_k^s - q_k^s(n)$ denote the net trade in k of a player $n \in N$ by trading at post Υ_k^s. We also define $B^s_{-\lambda,k} = \int_{N \setminus \{\lambda\}} b_k^s(n) d\mu$, and $Q^s_{-\lambda,k} = \int_{N \setminus \{\lambda\}} q_k^s(n) d\mu$.

Consumption allocations, $x_{h,k}\,(b(h), q(h), B_{-h}, Q_{-h}) \equiv x_{h,k}$, for commodities $k \in \{m\} \cup K$, to pure traders μ-$a.e$, $h \in N$, are determined as follows:

$$x_{h,k}=\begin{cases} e_k(h)+\sum_{s=1}^{|P|}\Big(b_k^s(h)\cdot\frac{Q_k^s}{B_k^s}-q_k^s(h)\Big) & \text{if } k\neq m;\\ e_k(h)+\sum_{k=1}^{\infty}\sum_{s=1}^{|P|}\Big(q_k^s(h)\cdot\frac{B_k^s}{Q_k^s}-b_k^s(h)\Big)\cdot\big(1+t_h^{s,k}c^k\big) & \text{if } k=m,\end{cases}\tag{1}$$

where $t_h^{s,k}:\mathbb{R}\supset z_k^s(h)\to\{-1,+1\}$, c^k is the proportional service charge payable (per unit of monetary net trade) at all trading posts for k, and we adopt the market game convention $\frac{0}{0}=0$ whenever it appears in the rule above. We will only write $t_h^{s,k}$ to denote $t_h^{s,k}(z_k^s(h))$. The second expression in the allocation rule above also includes the total premia payable to post owners, in terms of commodity m. The premium payable at a trading post by an individual h depends on the difference, in that very market itself, between h's sales receipts and his bid placed. In this light, we stipulate that $t_h^{s,k}=+1$ if $z_k^s(h)>0$, and $t_h^{s,k}=-1$ if $z_k^s(h)<0$ (such that agent h actually *pays* a premium). If $z_k^s(h)=0$, then, as in Toraubally (2018), we use the following rule:

$$t_h^{s,k}=\begin{cases}+1 & \text{if } \exists\theta\in\mathcal{N},\ \text{where } \mu(\theta\cap N_0)>0,\ \text{such that } \mu\text{-}a.e, \varsigma\in(\theta\cap N_0), z_k^s(\varsigma)\geq 0;\\ -1 & \text{otherwise.}\end{cases}$$

Consumption allocations, $x_{i,k}\left(b(i),q(i),B_{-i},Q_{-i}\right)\equiv x_{i,k}$, for $k=m,1,2,\ldots$, to any post owner $i\in A$, are determined as:

$$x_{i,k}=\begin{cases} e_k(i)+\sum_{s=1}^{|P|}\Big(b_k^s(i)\cdot\frac{Q_k^s}{B_k^s}-q_k^s(i)\Big) & \text{if } k\neq m;\\ e_k(i)-\sum_{k=1}^{\infty}c^k\cdot\Big(\int_N t_n^{i,k}q_k^i(n)d\mu\cdot\frac{B_k^i}{Q_k^i}-\int_N t_n^{i,k}b_k^i(n)d\mu\Big) & \\ \qquad +\sum_{k=1}^{\infty}\sum_{s=1}^{|P|}\Big(q_k^s(i)\cdot\frac{B_k^s}{Q_k^s}-b_k^s(i)\Big)\cdot\big(1+t_i^{s,k}c^k\big) & \text{if } k=m.\end{cases}\tag{2}$$

The conditions on $t_i^{s,k}$ are as for pure traders above. The second expression in the above rule not only includes total premia receivable at posts i owns, but also amounts that i needs to pay his fellow post owners. We impose that the total endowment and allocation of any commodity $k\in\{m\}\cup K$ in the economy be such that $0<\int_N x_{n,k}d\mu\leq\int_N e_k(n)d\mu<\infty$, with $\sum_{k\in\{m\}\cup K}\int_N e_k(n)d\mu<\infty$. This technical restriction guarantees the mapping $x(\cdot)$ is Gâteaux differentiable (in b and q).

Based on the above construct, we may explicitly derive the premium payable at a post Υ_k^s by any agent $n\in N$ as $-c^k t_n^{s,k}(q_k^s(n)\cdot p_k^s-b_k^s(n))$, such that his/her total premia payable across all markets for all goods may then be defined as $\Lambda(n)=\sum_{k=1}^{\infty}\sum_{s=1}^{|P|}-c^k t_n^{s,k}(q_k^s(n)\cdot p_k^s-b_k^s(n))$.

An equilibrium for this model is defined as a profile of agents' buy-and-sell decisions across all trading posts and commodities $(b,q)\in Gr(S)$ which forms a Nash equilibrium (N.E). At an equilibrium with positive bids and offers, agents can be viewed as solving the following problem:

$$\max_{(b(n),q(n))\in S(n)}\Big\{u_n\Big(\big(x_{n,k}\big(b(n),q(n),B_{-n},Q_{-n}\big)\big)_{k=1}^{\infty},x_{n,m}\big(b(n),q(n),B_{-n},Q_{-n}\big)\Big)\Big\}.\tag{3}$$

Using Propositions 1-3, we will next derive properties of equilibria for the above economy with at least two active posts (markets) per commodity. A post is *active* if price is *positive* and there is trade, i.e., the commodity in question actually changes hands.

2.2 Characterisation of Equilibria

Propositions 4 and 5 characterise equilibrium prices for a commodity between pairs of markets. Theorem 1 captures the failure of the LOP.

PROPOSITION 4. At an N.E with positive bids and offers, and no binding liquidity and offer constraints, the prices for any commodity $k \in K$ between any two active trading posts $\Upsilon^i_k, \Upsilon^j_k$ should satisfy the following condition:

$$\text{For any pure trader } h \in N: \qquad \left(p^i_k\right)^2 = \frac{B^i_{-h,k} Q^j_{-h,k}\left(1 + t^{j,k}_h c^k\right)}{Q^i_{-h,k} B^j_{-h,k}\left(1 + t^{i,k}_h c^k\right)} \left(p^j_k\right)^2.$$

PROPOSITION 5. Fix any commodity $k \in K$. Consider an active post owned by an agent $i \in A$, and another post owned by some $j \in A, i \neq j$. Then, at an N.E with positive bids and offers, and no binding liquidity and offer constraints, the prices at posts $\Upsilon^i_k, \Upsilon^j_k$ should satisfy the following condition for i:

$$\text{For } i: \quad \left(p^i_k\right)^2 = \frac{B^i_{-i,k} Q^j_{-i,k}\left(1 + t^{j,k}_i c^k\right)}{\left[Q^i_{-i,k} + c^k \cdot \left(\int_{N\setminus\{i\}} t^{i,k}_n q^i_k(n) d\mu\right)\right] B^j_{-i,k}} \left(p^j_k\right)^2.$$

THEOREM 1. Consider any good $k \in K$. If at equilibrium $\exists \theta \in \mathcal{N}$, such that $\mu(\theta \cap N_0) > 0$, and μ-$a.e, \varsigma \in (\theta \cap N_0)$, $z^i_k(\varsigma) \geq 0$, $z^j_k(\varsigma) < 0$, $i, j \in A, i \neq j$, then $p^i_k \neq p^j_k$.

PROOF: For any small pure trader $\varsigma \in N_0$, we have that $B^i_{-\varsigma,k} = \int_{N\setminus\{\varsigma\}} b^i_k(n) = \int_N b^i_k(n) = B^i_k$, and $Q^i_{-\varsigma,k} = \int_{N\setminus\{\varsigma\}} q^i_k(n) = \int_N q^i_k(n) = Q^i_k$. Using this fact together with the formula in Proposition 4, the statement of the theorem then implies that for every $\varsigma \in (\theta \cap N_0)$, $\frac{p^i_k}{p^j_k} = \frac{1-c^k}{1+c^k} \neq 1$. □

3 The failure of the LOP with many commodities: an example

Let $(N, \mathcal{N}, \mu)$ be a measure space of agents as defined in Section 2. Consider an economy where $N = N_0 \cup A \cup C$, $N_0 = (0, 1]$, $A = \{2, 3\}$, and $C = \{4, 5, \ldots\}$, where agents 2 and 3 are large post owners. We denote agent 2 by i, agent 3 by j, and a representative agent in C by γ. The set of commodities is $K = K_O \cup K_E$, where $K_O = \{1, 3, 5, \ldots\}$, $K_E = \{2, 4, 6, \ldots\}$, and i and j each own a single post for each commodity. The capacity of each post is $\mu(N) = 4$. The service charges allocated to i and j for commodities $a \in K_O$ and $b \in K_E$ are $(c^a, c^b) = (\frac{1}{6}, \frac{2}{17})$.

The preferences of individuals are represented by the following utility functions:[10]

[10] It is easily verifiable that the utility functions below satisfy Assumption (iii). Gâteaux-differentiability of $u(\cdot)$ may be verified by first taking limits, and using any appropriate test of convergence to establish existence.

$$
\begin{aligned}
&u_\varsigma(\mathbf{x}_\varsigma) = \textstyle\sum_{a\in K_O}(\frac{1}{2})^{a-1}\cdot 16.05\ln(x_{\varsigma,a}) + \sum_{b\in K_E}(\frac{1}{2})^{b-2}\cdot 17.00\ln(x_{\varsigma,b}) + 1.23\ln(x_{\varsigma,m}), \mu\text{-}a.e, \varsigma\in N_0,\\
&u_i(\mathbf{x}_i) = \textstyle\sum_{a\in K_O}(\frac{1}{2})^{a-1}\cdot 10.91\ln(x_{i,a}) + \sum_{b\in K_E}(\frac{1}{2})^{b-2}\cdot 11.00\ln(x_{i,b}) + 0.90\ln(x_{i,m}),\\
&u_j(\mathbf{x}_j) = \textstyle\sum_{a\in K_O}(\frac{1}{2})^{a-1}\cdot 18.28\ln(x_{j,a}) + \sum_{b\in K_E}(\frac{1}{2})^{b-2}\cdot 19.42\ln(x_{j,b}) + 1.37\ln(x_{j,m}),\\
&u_\gamma(\mathbf{x}_\gamma) = \textstyle\sum_{a\in K_O}(\frac{1}{2})^{a-1}\cdot 14.32\ln(x_{\gamma,a}) + \sum_{b\in K_E}(\frac{1}{2})^{b-2}\cdot 15.17\ln(x_{\gamma,b}) + 1.10\ln(x_{\gamma,m}), \forall\gamma\in C.
\end{aligned}
$$

The commodity endowments of the agents are as follows:

$$
\begin{aligned}
&a.e, \varsigma\in N_0, (e_a(\varsigma), e_b(\varsigma), e_m(\varsigma)) = (\tfrac{1}{2^{a-1}}\cdot 49.91, \tfrac{1}{2^{b-2}}\cdot 49.84, 54.50), a\in K_O, b\in K_E,\\
&(e_a(i), e_b(i), e_m(i)) = (\tfrac{1}{2^{a-1}}\cdot 50.11, \tfrac{1}{2^{b-2}}\cdot 50.19, 44.65), a\in K_O, b\in K_E,\\
&(e_a(j), e_b(j), e_m(j)) = (\tfrac{1}{2^{a-1}}\cdot 49.98, \tfrac{1}{2^{b-2}}\cdot 49.97, 50.85), a\in K_O, b\in K_E,\\
&\forall\gamma\in C, (e_a(\gamma), e_b(\gamma), e_m(\gamma)) = (\tfrac{1}{2^{a-1}}\cdot 50.00, \tfrac{1}{2^{b-2}}\cdot 50.00, 50.00), a\in K_O, b\in K_E.
\end{aligned}
$$

Now, consider *any* $a \in K_O$ and $b \in K_E$. It can be verified that the strategies below satisfy the conditions as in Theorem 1 and Propositions 1-5, therefore constituting an (unequal-price) N.E:

For any commodity $a \in K_O$:

$$
\begin{aligned}
&a.e, \varsigma\in N_0, (b^i_a(\varsigma), q^i_a(\varsigma), b^j_a(\varsigma), q^j_a(\varsigma)) = (\tfrac{1}{2^{a-1}}\cdot 1.000, \tfrac{1}{2^{a-1}}\cdot 0.000, \tfrac{1}{2^{a-1}}\cdot 0.000, \tfrac{1}{2^{a-1}}\cdot 0.001), a\in K_O,\\
&(b^i_a(i), q^i_a(i), b^j_a(i), q^j_a(i)) = (\tfrac{1}{2^{a-1}}\cdot 0.429, \tfrac{1}{2^{a-1}}\cdot 0.148, \tfrac{1}{2^{a-1}}\cdot 1.000, \tfrac{1}{2^{a-1}}\cdot 0.064), a\in K_O,\\
&(b^i_a(j), q^i_a(j), b^j_a(j), q^j_a(j)) = (\tfrac{1}{2^{a-1}}\cdot 9.000, \tfrac{1}{2^{a-1}}\cdot 0.787, \tfrac{1}{2^{a-1}}\cdot 0.091, \tfrac{1}{2^{a-1}}\cdot 0.004), a\in K_O,\\
&\forall\gamma\in C, (b^i_a(\gamma), q^i_a(\gamma), b^j_a(\gamma), q^j_a(\gamma)) = (\tfrac{1}{2^{a-1}}\cdot 8.529, \tfrac{1}{2^{a-1}}\cdot 0.765, \tfrac{1}{2^{a-1}}\cdot 0.008, \tfrac{1}{2^{a-1}}\cdot 0.001), a\in K_O.
\end{aligned}
$$

For the above strategies, the prevailing market-clearing prices are $p^i_a = 11.15$, and $p^j_a = 15.62$.

For any commodity $b \in K_E$:

$$
\begin{aligned}
&a.e, \varsigma\in N_0, (b^i_b(\varsigma), q^i_b(\varsigma), b^j_b(\varsigma), q^j_b(\varsigma)) = (\tfrac{1}{2^{b-2}}\cdot 2.000, \tfrac{1}{2^{b-2}}\cdot 0.000, \tfrac{1}{2^{b-2}}\cdot 0.000, \tfrac{1}{2^{b-2}}\cdot 0.001), b\in K_E,\\
&(b^i_b(i), q^i_b(i), b^j_b(i), q^j_b(i)) = (\tfrac{1}{2^{b-2}}\cdot 0.429, \tfrac{1}{2^{b-2}}\cdot 0.222, \tfrac{1}{2^{b-2}}\cdot 1.100, \tfrac{1}{2^{b-2}}\cdot 0.071), b\in K_E,\\
&(b^i_b(j), q^i_b(j), b^j_b(j), q^j_b(j)) = (\tfrac{1}{2^{b-2}}\cdot 9.045, \tfrac{1}{2^{b-2}}\cdot 0.709, \tfrac{1}{2^{b-2}}\cdot 0.100, \tfrac{1}{2^{b-2}}\cdot 0.005), b\in K_E,\\
&\forall\gamma\in C, (b^i_b(\gamma), q^i_b(\gamma), b^j_b(\gamma), q^j_b(\gamma)) = (\tfrac{1}{2^{b-2}}\cdot 9.794, \tfrac{1}{2^{b-2}}\cdot 0.794, \tfrac{1}{2^{b-2}}\cdot 0.008, \tfrac{1}{2^{b-2}}\cdot 0.001), b\in K_E.
\end{aligned}
$$

For the above strategies, the prevailing market-clearing prices are $p^i_b = 12.33$, and $p^j_b = 15.62$.

Based on the above profile of strategies, the final consumption of all agents μ-$a.e, n \in N$, is $(x_{n,a}, x_{n,b}, x_{n,m}) = (50/2^{a-1}, 50/2^{b-2}, 50)$, $\forall a \in K_O$, $\forall b \in K_E$.

4 Discussion and conclusion

The intuition behind the failure of the LOP is as follows. Consider any commodity $k \in K$, and, e.g., a large agent who wishes to shift his orders so as to buy more from the cheaper market (call it $\mathscr{C}$), and sell more in the expensive market ($\mathscr{E}$). In doing so, he increases his bids and offers in $\mathscr{C}$ and $\mathscr{E}$, respectively. But what this merely achieves is increase the price in $\mathscr{C}$, and decrease it in $\mathscr{E}$—i.e., for any large agent, the marginal price of k is not equal to its average price. Additionally, this change in the market-clearing prices can be so drastic as to cause our trader to end up with a resultant allocation which actually makes him worse off. Thus, he does not deviate. But what about small agents who cannot affect prices? Their efforts to profit from any price disparity are hindered by the service charge, even though it is the same across markets for the same commodity. The key lies in realising that traders are charged on their net, as opposed to gross, trades. In this vein, let us picture a small agent who is a net buyer in $\mathscr{E}$, and a net seller in $\mathscr{C}$. Assume that this agent now wishes to shift her orders such that whatever she had bidden in $\mathscr{E}$ is now bidden in $\mathscr{C}$, and whatever was sold in $\mathscr{C}$ is now sold in $\mathscr{E}$. What happens is that while initially she was charged only on the *difference* between her purchases and sales in each market (see the structure of net trades in Theorem 1), with this new configuration of strategies, she is now charged on the full amounts of her bids and sales in each market. Thus, whatever was gained by 'gaming' the prices is, *at best*, more than outweighed by the increase in premia payable. So our small agent stays put.

Note that the failure of the LOP is not due only to the presence of large players. In fact, so long as agents have the financial means to arbitrage prices, conventional wisdom dictates that the LOP should obtain when the number of players tends to infinity, whether or not the corresponding limit economies contain large players who can influence prices. Verily, the set of equilibria in which the LOP holds is a *superset* of perfectly-competitive equilibrium outcomes. As such, it is necessary, but *not sufficient*, for the LOP to obtain for there to be perfect competition.

In this paper, we have presented a framework with infinitely many commodities in which the LOP fails. This important result is new. Moreover, the example that we have produced in this work is easily understandable, yet involved: it is indeed not trivial to explicitly derive equilibrium allocations which satisfy, for infinitely many agents, the marginal rate of substitution of commodity k for m, for infinitely many such k! This example is also robust as far as utility functions and initial endowments are concerned: any collection of such which satisfy the relevant first-order necessary and sufficient conditions (see Appendix B) will do.

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